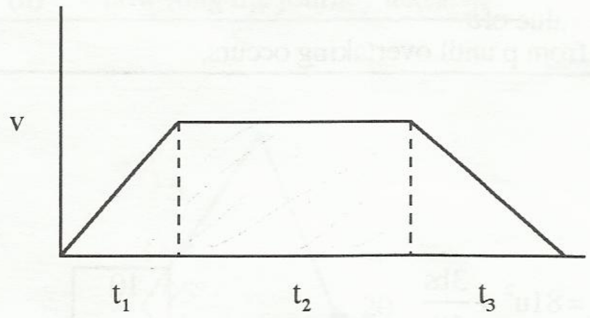
1998

1 (a) A train accelerates uniformly from rest to a speed v m/s. It continues at this constant speed for a period of time and then decelerates uniformly to rest. If the average speed for the whole journey is $\frac{5v}{6}$, find what fraction of the whole distance is described at constant speed.



5

distance $= (\frac{1}{2}t_1 + t_2 + \frac{1}{2}t_3)v$ 5

Average speed $= \dfrac{\text{distance}}{\text{time}}$

$$\frac{5v}{6} = \frac{(\frac{1}{2}t_1 + t_2 + \frac{1}{2}t_3)v}{t_1 + t_2 + t_3}$$ 5

$\frac{2}{3}(t_1 + t_2 + t_3) = t_2$

fraction $= \dfrac{t_2 v}{(\frac{1}{2}t_1 + t_2 + \frac{1}{2}t_3)v}$ Area 2

$$= \frac{\frac{2}{3}(t_1 + t_2 + t_3)}{\frac{5}{6}(t_1 + t_2 + t_3)} = \frac{4}{5}$$ 10

25

1998

1 (b) Car A, moving with uniform acceleration $\frac{3b}{20}$ m/s^2 passes a point p with speed $9u$. m/s. Three seconds later Car B, moving with uniform acceleration $\frac{2b}{9}$ m/s^2 passes the same point with speed $5u$. m/s. B overtakes A when their speeds are 6.5 m/s and 5.4 m/s respectively.

Find (i) the value of u and the value of b
(ii) the distance travelled from p until overtaking occurs.

(i) Car A $(5.4)^2 = 81u^2 + \frac{3bs}{10}$ 10

Car B $(6.5)^2 = 25u^2 + \frac{4bs}{9}$ 10

$$291.6 = 810u^2 + 3bs$$

$$380.25 = 225u^2 + 4bs$$

$$\Rightarrow u = 0.1 \text{ m/s}$$

Car A $5.4 = 0.9 + \frac{3bt}{20}$

Car B $6.5 = 0.5 + \frac{2b(t-3)}{9}$

$$\Rightarrow b = 1 \quad (\text{and} \quad t = 30)$$

(ii) Car A $291.6 = 810(0.1)^2 + 3(1)s$

$$\Rightarrow s = 94.5\,m$$ 5

25

1998

2 (a)	The driver of a speedboat travelling in a straight line at 20 m/s wishes to intercept a yacht travelling at 5 m/s in a direction 40° East of North. Intially the speedboat is positioned 5 km South-East of the yacht. Find (i) the direction of the speedboat if it intercepts the yacht (ii) how long the journey takes.

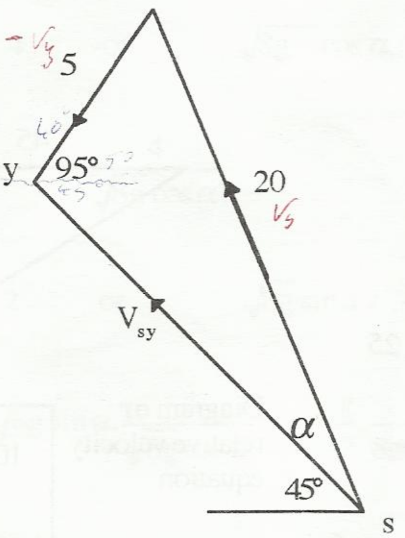


Diagram or vector approach — 10

For interception to occur V_{sy} must be in the direction sy — 5

$$\frac{\sin\alpha}{5} = \frac{\sin 95}{20}$$

$$\alpha = 14.42°$$ 5

$\therefore$ direction is W 59.42° N

$$\frac{V_{sy}}{\sin 70.8°} = \frac{20}{\sin 95°}$$

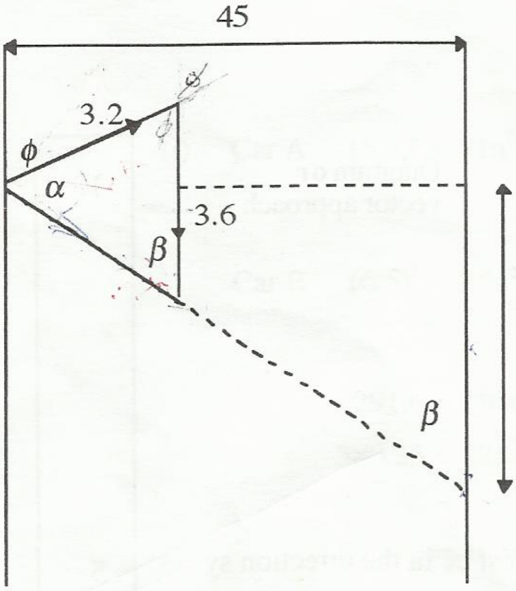
$$V_{sy} = 18.93 \text{ m/s}$$

$$\text{time} = \frac{5000}{V_{sy}} = 264.13 \text{ s}$$ 5

25

1998 2(b)

2 (b)	A man wishes to row a boat across a river to reach a point on the opposite bank that is 25 m downstream from his starting point. The man can row the boat at 3.2 m/s in still water. The river is 45 m wide and flows uniformly at 3.6 m/s. Find	
	(i)	the two possible directions in which the man could steer the boat
	(ii)	the respective crossing times.



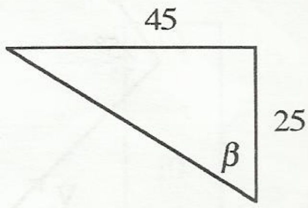


Diagram **or** relative velocity equation — 10

$\tan\beta = \dfrac{45}{25} \quad \Rightarrow \quad \beta = 60.95°$ — 5

$\dfrac{\sin\alpha}{3.6} = \dfrac{\sin 60.95}{3.2}$

$\Rightarrow \quad \alpha = 79.55° \quad \text{or} \quad 100.45°$

$\Rightarrow \quad \phi = 39.5° \quad \text{or} \quad 18.6°$ — 5

$\text{time}_1 = \dfrac{45}{3.2\sin 39.5°} = 22.1 \text{ s}$

$\text{time}_2 = \dfrac{45}{3.2\sin 18.6°} = 44.1 \text{ s}$ — 5

25

1998

3 (a) A football is kicked from a spot on level ground with a velocity of $\sqrt{8g}$ m/s and strikes a vertical wall 4 m away at a point 2 m above the ground. Find the two possible angles of projection.

$$\vec{r} = (\sqrt{8g}\cos\alpha.t)\,\vec{i} + (\sqrt{8g}\sin\alpha.t - \tfrac{1}{2}gt^2)\,\vec{j} \qquad 5$$

$$r_i = 4 \quad \text{or} \quad \sqrt{8g}\cos\alpha.t = 4 \qquad 5$$

$$\Rightarrow t = \frac{4}{\sqrt{8g}\cos\alpha} \qquad 5$$

$$r_j = 2 \quad \text{or} \quad \sqrt{8g}\sin\alpha.t - \tfrac{1}{2}gt^2 = 2 \qquad 5$$

$$\Rightarrow \sqrt{8g}\sin\alpha.\frac{4}{\sqrt{8g}\cos\alpha} - \frac{g}{2}.\frac{16}{8g\cos^2\alpha} = 2$$

$$4\tan\alpha - (1 + \tan^2\alpha) = 2$$

$$\tan^2\alpha - 4\tan\alpha + 3 = 0$$

$$\tan\alpha = 1 \quad \text{or} \quad \tan\alpha = 3 \qquad 5$$

25

$\alpha = 45$ / $71°33'$

1998

3 (b)	A particle is projected down a slope which is inclined at 45° to the horizontal. The particle is projected from a point on the slope and has an initial velocity of $7\sqrt{2}$ m/s at an angle α to the inclined plane. Find the value of α if (i) the particle first hits the slope after 2 seconds (ii) the landing angle with the slope is $\tan^{-1}\left(\frac{1}{3}\right)$

$$\vec{r} = (7\sqrt{2}\cos\alpha.t + \tfrac{1}{2}g\sin 45.t^2)\,\vec{i} + (7\sqrt{2}\sin\alpha.t - \tfrac{1}{2}g\cos 45.t^2)\,\vec{j}$$

$$\vec{v} = (7\sqrt{2}\cos\alpha + g\sin 45.t)\,\vec{i} + (7\sqrt{2}\sin\alpha - g\cos 45.t)\,\vec{j}$$

Particle hits plane $\Rightarrow r_j = 0 \Rightarrow t = \dfrac{14\sqrt{2}\sin\alpha}{g\cos 45}$ 5

$$\Rightarrow 2 = \frac{28\sin\alpha}{g}$$ 5

$$\sin\alpha = \frac{g}{14} \quad \text{or} \quad \alpha = 44.43°$$ 5

(ii) $\tan(\text{landing angle}) = \dfrac{-V_j}{V_i}$ 5

$$\frac{1}{3} = \frac{\frac{g}{\sqrt{2}}t - 7\sqrt{2}\sin\alpha}{7\sqrt{2}\cos\alpha + \frac{g}{\sqrt{2}}t}$$

$$7\sqrt{2}\cos\alpha + \frac{g}{\sqrt{2}}\frac{28\sin\alpha}{g} = \frac{3g}{\sqrt{2}}\frac{28\sin\alpha}{g} - 21\sqrt{2}\sin\alpha$$

$$\tan\alpha = 1 \quad \text{or} \quad \alpha = 45°$$ 5

25

cancel g's, Multiply top line by $\sqrt{2}$

$14\cos\alpha + 28\sin\alpha = 84\sin\alpha - 42\sin\alpha$

$\Rightarrow \cos\alpha = \sin\alpha$

$\Rightarrow \tan\alpha = 1$

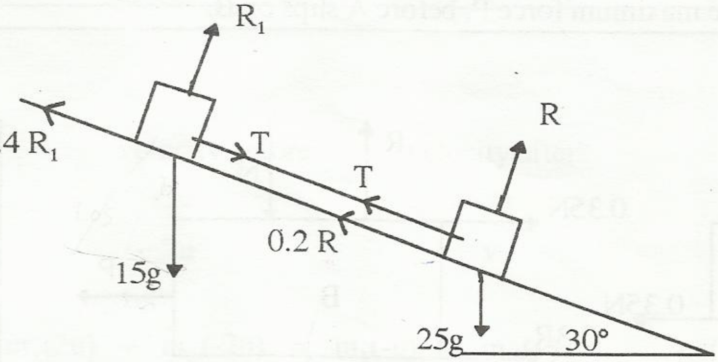
$\Rightarrow \alpha = 45°$

1998

4 (a) Blocks A and B, of mass 15 kg and 25 kg, respectively, are connected by a light, inextensible string as shown in the diagram. The coefficients of friction are 0.4 for block A and 0.2 for block B. The blocks move down the plane which is inclined at 30° to the horizontal. Find

(i) the acceleration of block B

(ii) the tension in the string.



10

Block A $\quad R_1 = 15g\cos30° = \dfrac{15g\sqrt{3}}{2}$ or 127.306

$$T + 15g\sin30° - 0.4\left(\frac{15g\sqrt{3}}{2}\right) = 15a \qquad \text{eq (1)}$$ 5

Block B $\quad R = 25g\cos30° = \dfrac{25g\sqrt{3}}{2}$ or 212.176

$$25g\sin30° - T - 0.2\left(\frac{25g\sqrt{3}}{2}\right) = 25a \qquad \text{eq (2)}$$ 5

Add equations (1) and (2) $\Rightarrow$ $20g - \dfrac{11g\sqrt{3}}{2} = 40a$

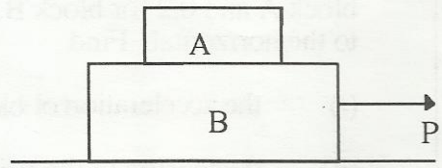
$\Rightarrow$ $a = 2.566\ \text{ms}^{-2}$ or $\dfrac{40g - 11\sqrt{3}\,g}{80}$ 5

From equation (1) $\quad T = 15(2.566) + 0.4(127.306) - 73.5$

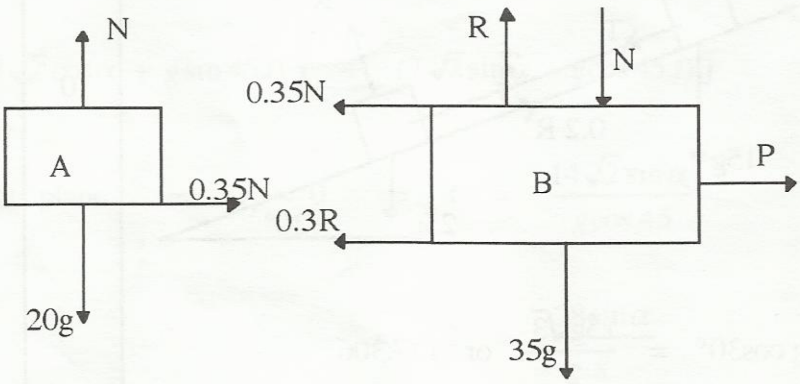
$\Rightarrow$ $T = 15.91\ \text{N}$ or $\dfrac{15g\sqrt{3}}{16}$ 5 | 30

1998

4 (b) The two blocks shown in the diagram are at rest on a horizontal surface when a force P is applied to block B. Blocks A and B have masses 20 kg and 35 kg, respectively. The coefficient of friction between the two blocks is 0.35 and the coefficient of friction between the horizontal surface and block B is 0.3.



Determine the maximum force P, before A slips on B.



frictional force 5

Block A vert. $N = 20g$ 5

horiz Force = mass x acceleration

$0.35(20g) = 20a$

$\Rightarrow a = 0.35g$

Block B vert. $R = N + 35g$ 5

horiz Force = mass x acceleration

$P - 0.35(N) - 0.3(R) = 35a$

$P - 0.35(20g) - 0.3(20g + 35g) = 35(0.35g)$

$\Rightarrow P = 35.75g$ or 350.35 N 5

20

1998

5 (a) Two smooth spheres A and B have masses m_1 and m_2, respectively. They are moving towards each other along the same horizontal line each with speed $2u$. After collision both spheres reverse their original directions of motion and A now travels with speed u.

(i) Show that $3m_1 > 2m_2$.

(ii) Find an expression for e, the coefficient of restitution, and hence or otherwise show that $3m_1 \leq 5m_2$.

	mass	velocity before	velocity after
A	m_1	$2u$	$-u$
B	m_2	$-2u$	v

P.C.M. $\quad m_1(2u) + m_2(-2u) = m_1(-u) + m_2(v) \quad$ eq(1) $\quad$ 10

N.E.L. $\quad v - (-u) = -e(-2u - 2u) \quad$ eq(2) $\quad$ 10

(i) From eq(1) $\quad v = \dfrac{3m_1u - 2m_2u}{m_2}$

$v > 0 \quad \Rightarrow \quad 3m_1 > 2m_2 \quad$ 5

(ii) From eq(2) $\quad e = \dfrac{v + u}{4u}$

$e \leq 1 \quad \Rightarrow \quad v + u \leq 4u$

$\Rightarrow \quad v \leq 3u$

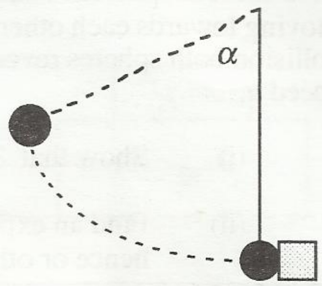
$\Rightarrow \quad \dfrac{3m_1u - 2m_2u}{m_2} \leq 3u$

$\Rightarrow \quad 3m_1 \leq 5m_2 \quad$ 5 $\quad$ 30

1998

5 (b) A sphere of mass 4 kg is released from rest when $\alpha = 60°$. It swings down and strikes a 7 kg box B when the string is vertical. The distance from the point of support to the centre of the sphere is one metre and the coefficient of restitution for the collision is $\frac{3}{4}$.

Calculate the speed of the box immediately after the impact if the box is free to move.



Gain in K.E. = Loss in P.E.

$\frac{1}{2}(4)v^2 = 4g(1 - 1.\cos 60)$

$\Rightarrow \quad v = \sqrt{g}$

10

mass	velocity before	velocity after
4	$\sqrt{g}$	v_1
7	0	v_2

P.C.M. $\quad 4\sqrt{g} + 0 = 4v_1 + 7v_2$

N.E.L. $\quad v_1 - v_2 = -\frac{3}{4}(\sqrt{g} - 0)$

5

$$4v_1 + 7v_2 = 4\sqrt{g}$$

$$4v_1 - 4v_2 = -3\sqrt{g}$$

$$\Rightarrow \quad v_2 = \frac{7\sqrt{g}}{11} \quad \text{or} \quad 1.99$$

5

20

'98

6 (a)	Define Simple Harmonic Motion.
	The distance, x, of a particle from a fixed point, o, is given by $x = 7\sin\omega t + 24\cos\omega t$, ω being a constant.
	(i) Show that the particle is describing simple harmonic motion about o.
	(ii) Calculate the amplitude of the motion.

The motion of a particle is simple harmonic motion if its acceleration towards a particular point is proportional to its displacement from that point. 10

(i) $x = 7\sin\omega t + 24\cos\omega t$

$$\frac{dx}{dt} = 7\omega\cos\omega t - 24\omega\sin\omega t$$ 5

$$\frac{d^2x}{dt^2} = -7\omega^2\sin\omega t - 24\omega^2\cos\omega t$$ 5

$$= -\omega^2 x$$ 5

$\therefore$ S.H.M. about $x = 0$

(ii) amplitude $= \sqrt{7^2 + 24^2}$

$= 25$ 5

30

98

6 (b)	An elastic string of natural length one metre is extended 20 cm by a particle attached to its end and hanging freely. The particle is then pulled down a further distance of 40 cm and released. (i) Show that the particle moves with simple harmonic motion when the string is taut (ii) Find the height above the equilibrium position to which the particle will rise.

(i) Equilibrium position

$$T_0 = mg$$

$$k(0.2) = mg$$

$$\Rightarrow \quad k = 5mg \quad \text{or} \quad 49m$$ 5

Displaced position

Force in direction of x increasing $= mg - k(0.2 + x)$ 5

$= mg - mg - 5mgx$

$= -5mgx$

acceleration $= -5gx$ 5

$= -49x$

$\Rightarrow$ S.H.M. about $x = 0$ with $\omega = 7$

(ii) Find velocity of particle when string is 1 m long

$$v = \omega\sqrt{a^2 - x^2}$$

$$= 7\sqrt{(0.4)^2 - (0.2)^2}$$

$$= 7\sqrt{0.12} \quad \text{or} \quad 2.43$$

Find distance

$$v^2 = u^2 + 2as$$

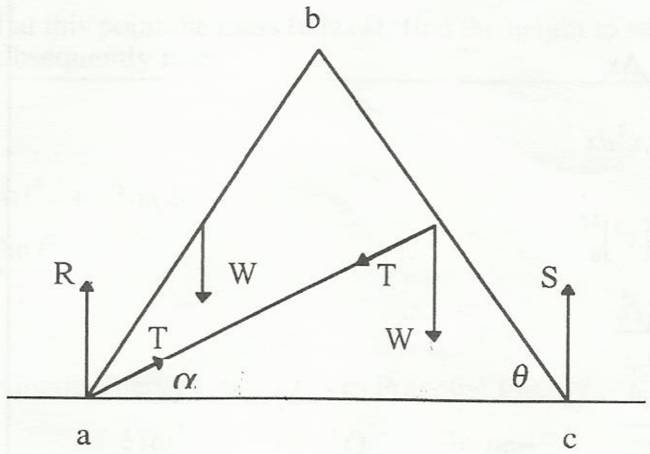
$$0 = 49(0.12) + 2(-9.8)s$$

$\Rightarrow s = 0.3 \Rightarrow$ 0.5 m above the equilibrium position 5

20

1998

7	Two equal uniform rods [ab] and [bc], each of weight W, are freely jointed at b. An inextensible string connects a to the midpoint of [bc]. When the string is taut the angle bca is θ. The rods are placed in a vertical plane with a and c on a smooth horizontal surface. Prove that the tension in the string is $\dfrac{W}{4}\sqrt{1+9\cot^2\theta}$.



10

Resolve vertically $\quad R + S = 2W$ $\quad$ 5

Moments about a for system $\quad W(1) + W(3) = S(4)$ $\quad$ 5

$\Rightarrow \quad S = W \quad$ and $\quad R = W$ $\quad$ 5

Moments about b for ba

ACW ACW CW CW

$$T\cos\alpha.2\ell\sin\theta + W.\ell\cos\theta = T\sin\alpha.2\ell\cos\theta + R.2\ell\cos\theta$$ 5,5

$$2T\cos\alpha.\tan\theta + W = 2T\sin\alpha + 2W \quad (\because R = W)$$

$$\Rightarrow \quad T\cos\alpha.\tan\theta = \frac{W}{2} + T\sin\alpha \quad \text{.......... eq (1)}$$

Moments about b for bc

CW CW CW ACW

$$T\cos\alpha.\ell\sin\theta + T\sin\alpha.\ell\cos\theta + W.\ell\cos\theta = S.2\ell\cos\theta$$ 5,5

$$T\cos\alpha.\tan\theta + T\sin\alpha + W = 2W \quad (\because S = W)$$

$$\Rightarrow \quad T\cos\alpha.\tan\theta = W - T\sin\alpha \quad \text{.......... eq (2)}$$

Solve equations (1) and (2)

$$T\sin\alpha = \frac{W}{4} \quad \text{and} \quad T\cos\alpha = \frac{3W}{4\tan\theta}$$

$$\Rightarrow \quad T^2\sin^2\alpha + T^2\cos^2\alpha = \frac{W^2}{16} + \frac{9W^2}{16\tan^2\theta}$$

$$\Rightarrow \quad T = \frac{W}{4}\sqrt{1 + 9\cot^2\theta}$$ 5

50

1998

8 (a)	Prove that the moment of inertia of a uniform rod [ab] of mass m and length 2ℓ about an axis through a, perpendicular to the rod, is $\frac{4}{3}m\ell^2$.

Let m_1 = mass per unit length

Mass of rod $m = 2m_1\ell$

Consider an element of the rod of width Δx, a distance x from the axis.

Mass of the element $= m_1\Delta x$ 5

Moment of inertia $= \int_0^{2\ell} m_1 x^2 dx$ 5

$= \frac{m_1}{3}\left[x^3\right]_0^{2\ell}$ 5

$= \frac{8m_1\ell^3}{3}$

$= \frac{4m\ell^2}{3}$ 5

20

8 (b)	A lamina is rotating with angular velocity ω about an axis perpendicular to its plane. If the moment of inertia of the lamina about the axis is I, prove that the kinetic energy is $\frac{1}{2}I\omega^2$.

Consider a particle of the body of mass m, a distance r from the axis.

Kinetic Energy of particle $= \frac{1}{2}mv^2$

$= \frac{1}{2}mr^2\omega^2$ 5

Kinetic Energy of the lamina $= \sum \frac{1}{2}mr^2\omega^2$

$= \frac{1}{2}\omega^2 \sum mr^2$

$= \frac{1}{2}I\omega^2$ 5

10

1998

8 (c)	A uniform rod [ab], of mass m and length 2ℓ, is free to rotate in a vertical plane about a fixed horizontal axis at a, with a particle of mass $3m$ attached to the rod at b. The system is released from rest with the rod vertical and the end b above a.
	(i) Show that the angular velocity of the rod when next it is vertical is $\sqrt{\dfrac{21g}{10\ell}}$.
	(ii) If at this point the mass falls off, find the height to which the end b subsequently rises.

$$I = \tfrac{4}{3}m\ell^2 + 3m(2\ell)^2$$ 5

$$= \frac{40m\ell^2}{3}$$ 5

Take bottom as base

Gain in Kinetic Energy = Loss in Potential Energy

$$\tfrac{1}{2}I\omega^2 = mg(2\ell) + (3m)g(4\ell)$$

$$\frac{20m\ell^2\omega^2}{3} = 14mg\ell$$

$$\Rightarrow \quad \omega = \sqrt{\frac{21g}{10\ell}}$$ 5

(ii) Loss in Kinetic Energy = Gain in Potential Energy

$$mgl + \quad \tfrac{1}{2}\left(\frac{4}{3}m\ell^2\right)\left(\frac{21g}{10\ell}\right) = mgh$$

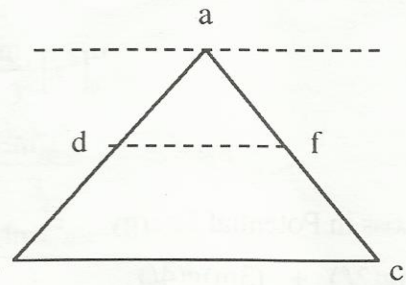
$$\Rightarrow \quad h = \frac{7\ell}{5}$$

End b will rise a distance $2h = \dfrac{14\ell}{5}$ 5

20

18

9 (a)	A triangular lamina abc is immersed in a vertical position in water with its vertex a at the surface and its base [bc] parallel to the surface.
(i)	If $|bc| = 10$ cm and the height of the triangle is 7.5 cm, find the thrust on abc due to the water.
(ii)	If d and f are the midpoints of [ab], [ac] respectively, find the ratio $$\frac{\text{thrust on adf}}{\text{thrust on dbcf}}$$



$$T_{abc} = \text{Pressure x Area}$$

$$= \rho g\left(\tfrac{2}{3} \text{ x } 0.075\right)\left\{\tfrac{1}{2}(0.1)(0.075)\right\}$$

$$= 0.1875g \quad \text{or} \quad 1.8375$$ 5

$$T_{adf} = \text{Pressure x Area}$$

$$= \rho g\left(\frac{2}{3} \text{ x } \frac{0.075}{2}\right)\left\{\tfrac{1}{2}(0.05)\left(\frac{0.075}{2}\right)\right\}$$

$$= 0.0234375g \quad \text{or} \quad 0.2296875$$ 5

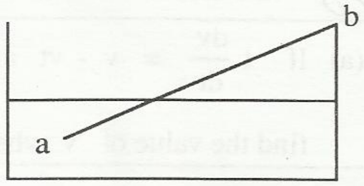
$$\Rightarrow \quad T_{dbcf} = 0.1640625g \quad \text{or} \quad 1.6078125$$ 5

$$\Rightarrow \quad \frac{T_{adf}}{T_{dbcf}} = \frac{0.2296875}{1.6078125} \quad \text{or} \quad \frac{1}{7} \quad \text{or } 1.43$$ 5

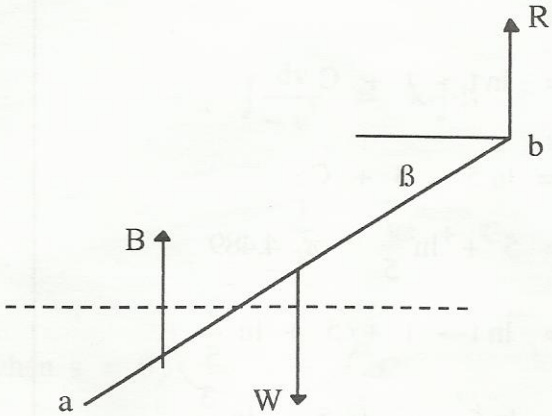
20

'98

9 (b) A thin uniform rod [ab] of length ℓ and relative density s is in equilibrium in an inclined position with the end a immersed in a container of water and the end b supported on the edge of the container.



Show that the length of the immersed part of the rod is $\ell(1 - \sqrt{1 - s})$.



10

Let x = length of immersed part

Resolve vertically $\quad R + B = W$ 5

$$B = \frac{W_I s_L}{s} = \frac{xW}{\ell s}$$

Take moments about b

$$B\left(\ell - \frac{x}{2}\right)\cos\beta = W\,\tfrac{1}{2}\ell\cos\beta$$ 5,5

$$\frac{xW}{\ell s}\left(\ell - \frac{x}{2}\right) = W\,\tfrac{1}{2}\ell$$

$$\Rightarrow \quad x^2 - 2\ell x + \ell^2 s = 0$$

$$\Rightarrow \quad x = \ell(1 - \sqrt{1 - s}) \qquad \text{as } x < \ell$$ 5

30

1998

10 (a) If $t\dfrac{dv}{dt} = v - vt$ and $v = 3$ when $t = 5$,

find the value of v when $t = 6$.

$$\int \frac{dv}{v} = \int \left(\frac{1 - t}{t}\right) dt \qquad 5$$

$$\ln v = \ln t - t + C \qquad 5$$

$v = 3$ when $t = 5$ $\Rightarrow$ $\ln 3 = \ln 5 - 5 + C$ $\qquad 5$

$$\Rightarrow \quad C = 5 + \ln\frac{3}{5} \quad \text{or} \quad 4.489$$

$$\therefore \quad \ln v = \ln t - t + 5 + \ln\frac{3}{5}$$

Find v when $t = 6$ $\Rightarrow$ $\ln v = \overset{1.79}{\ln 6} - 6 + 5 + \ln\frac{3}{5}$

$$\Rightarrow \quad \ln v = \ln\frac{18}{5} - 1 \quad \text{or} \quad 0.281$$

$$\Rightarrow \quad v = 1.32 \quad \text{or} \quad \frac{18}{5e} \quad \text{or} \quad e^{0.281} \qquad 5 \qquad 20$$

1998

10 (b)	A particle moves in a straight line. The initial speed is u and the retardation is kv^3, where v is the speed at the time t. If s is the distance travelled in time t, prove (i) $v = \dfrac{u}{1 + ksu}$ (ii) $t = \dfrac{ks^2}{2} + \dfrac{s}{u}$.

$$v\frac{dv}{ds} = -kv^3$$

$$\int \frac{dv}{-v^2} = k\int ds$$ 5

$$\frac{1}{v} = ks + C$$ 5

$v = u$ when $s = 0$ $\Rightarrow$ $\dfrac{1}{u} = C$ 5

$$\therefore \quad \frac{1}{v} = ks + \frac{1}{u}$$

$$= \frac{ksu + 1}{u}$$

$$\Rightarrow \quad v = \frac{u}{ksu + 1}$$ 5 **20**

$$\frac{ds}{dt} = \frac{u}{ksu + 1}$$

$$\int (ksu + 1)\,ds = u\int dt$$ 5

$$\tfrac{1}{2}ks^2u + s = ut + A$$

$t = 0$ when $s = 0$ $\Rightarrow$ $0 = A$

$$\therefore \quad \tfrac{1}{2}ks^2u + s = ut$$

$$\Rightarrow \quad t = \frac{ks^2}{2} + \frac{s}{u}$$ 5 **10**